CALCULATION OF THE TEMPERATURE PROFILE OF A PERFORATED FIN

V. A. Kirpikov and I. I. Leifman

UDC 536.242:536.12

An approximate method is proposed for calculating the temperature profile of a thin perforated fin.

In connection with a study concerning one novel design of an effective heating surface, there arises the problem of determining the steady-state temperature field in a thin ($\delta/l \ll 1$) perforated fin (Fig. 1a). The essential operating feature of such a fin is that the longitudinal flow of the medium along its surface is accompanied by a transverse passage of that medium through the perforations (at constant values of ϑ_0, α, α_1, and λ).

As in [1], we will set up a differential equation of the temperature field in a fin element (Fig. 1b) in the one-dimensional approximation (the validity of such a representation when the holes are small, i.e., the ratio a/b is small, has been confirmed by subsequent experiments):

$$d\left[\lambda\delta h\frac{d\vartheta}{dx}\right]=2h\alpha\vartheta dx+\sqrt{1+(h')^2}\,\delta\alpha_1\vartheta dx. \tag{1}$$

After the left-hand side of Eq. (1) has been expanded, it can be reduced to a form not containing the first derivative [2]. The substitution $\vartheta = z/\sqrt{h}$ yields the following linear differential equation:

$$\frac{d^2z}{dx^2}+\Phi(x)z=0, \tag{2}$$

where

$$\Phi(x)=-\frac{1}{2}(\ln h)''-\frac{1}{4}[(\ln h)']^2-\frac{2\alpha}{\lambda\delta}\left[1+\frac{\alpha_1\delta}{2\alpha h}\sqrt{1+(h')^2}\right].$$

This is an equation of the Hill kind [3-6], because $\Phi(x)$ is an even periodic function with the period c. With $\Phi(x)$ expanded into a Fourier series and with a change of variables $t = 2\pi x/c$, this equation becomes

$$\frac{d^2z}{dt^2}+\left(R_0+\sum_{n=1}^{\infty}R_n\cos nt\right)z=0, \tag{3}$$

where

$$R_0=\frac{c^2}{4\pi^2}\cdot\frac{a_0}{2};\ R_n=\frac{c^2}{4\pi^2}a_n;\ a_n=\frac{4}{c}\int_0^{c/2}\Phi(x)\cos\frac{2\pi nx}{c}dx,\ n=0,1,2,\ldots$$

According to Flocke's theory [5], the general solution to Eq. (3) is

$$z=\theta_1e^{\mu t}\sum_{m=-\infty}^{\infty}C_me^{mti}+\theta_2e^{-\mu t}\sum_{m=-\infty}^{\infty}C_me^{-mti}. \tag{4}$$

After reconciling this solution with the boundary conditions

Institute of Chemical Apparatus Design, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 23, No. 2, pp. 316-321, August, 1972. Original article submitted December 30, 1971.

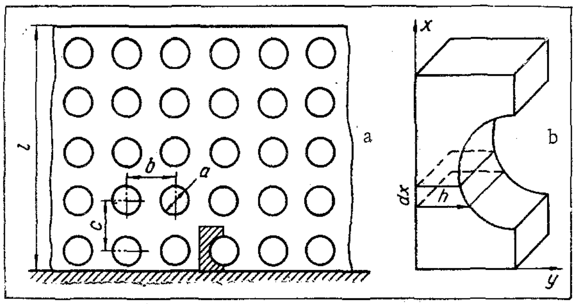

Fig. 1

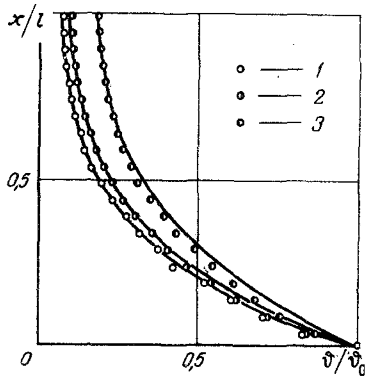

Fig. 2

Fig. 1. Schematic diagram of the fin (a) and its element (b).

Fig. 2. Temperature profile of a perforated fin, according to test data (points) and according to calculations (lines): 1) $\alpha = \alpha_1$; 2) α_1 reduced to one half; 3) α and α_1 reduced to one half relative to curve 1.

$$z = \vartheta_0 \sqrt{\frac{b}{2}} \text{ at } t = 0 \text{ and } \frac{dz}{dt} = 0 \text{ at } t = \frac{2\pi l}{c},$$

which correspond to the boundary conditions for Eq. (1) in its original form

$$\vartheta = \vartheta_0 \text{ at } x = 0 \text{ and } \frac{d\vartheta}{dx} = 0 \text{ at } x = l,$$

we obtain the temperature profile in a perforated fin insulated at the end:

$$\frac{\vartheta}{\vartheta_0} = \frac{\mathrm{ch}\left[2\pi \frac{l}{c} \mu \left(1 - \frac{x}{l}\right)\right]}{\mathrm{ch}\left(2\pi \frac{l}{c} \mu\right)} \sqrt{\frac{b}{2h(x)}} \; \frac{\sum\limits_{m=-\infty}^{\infty} C_m \cos \frac{2\pi m x}{c}}{\sum\limits_{m=-\infty}^{\infty} C_m}. \tag{5}$$

The values of μ and C_m can be found by solving Eq. (3) by Hill's method of the infinite determinant [5, 6].

The last two factors on the right-hand side of Eq. (5) constitute a periodic function with the period c. For values of x which are multiples of c this function becomes unity. One may state, therefore, that the monotonic temperature drop along the fin height – of particular interest here – is determined by the first factor only and that the other two factors represent spatial temperature fluctuations superposed on the fundamental profile. With sufficiently many rows of perforations along the fin height, the periodic fluctuations have obviously no significant effect on the shaping of the temperature profile. After differentiating Eq. (5), we obtain an expression for the thermal flux transmitted through a fin:

$$Q = -\lambda \frac{b}{2} \delta \left(\frac{d\vartheta}{dx}\right)_{x=0} = \lambda \frac{b}{2} \delta \frac{2\pi}{c} \mu \vartheta_0 \,\mathrm{th}\left(2\pi \frac{l}{c} \mu\right).$$

Thus, determining the fundamental temperature profile and the thermal flux involves the calculation of only one μ. For this purpose, according to [5, 6], one can use the transcendental equation

$$\sin^2\left(\frac{1}{2} i\mu\pi\right) = \Delta(0) \sin^2\left(\frac{1}{2}\pi \sqrt{R_0}\right), \tag{6}$$

where the value of the Hill determinant $\Delta(0)$ for $\mu = 0$ is estimated as

$$\Delta(0) \cong 1 + \frac{R_1^2 R_0}{4R_0(R_0 - 1)^2} - \frac{R_2^2}{4(R_0 - 1)^2} - \frac{R_1^2}{2(R_0 - 1) R_0}. \tag{7}$$

With small holes, the periodic deviations of $\Phi(x)$ from the mean values are small and R_1, R_2, . . . are small as compared to R_0. Therefore, for estimating purposes, we assume $\Delta(0) = 1$. Then Eq. (6) yields $\mu = \pm\sqrt{R_0}$ or

$$2\pi\frac{l}{c}\mu=\sqrt{\frac{2l^2}{c}\int_0^{c/2}\left\{\frac{1}{2}(\ln h)''+\frac{1}{4}[(\ln h)']^2\right\}dx+\frac{2}{c}\int_0^{c/2}\frac{2\alpha l^2}{\lambda\delta}\left[1+\frac{\alpha_1\delta}{2\alpha h}\sqrt{1+(h')^2}\right]dx}. \tag{8}$$

We will now show that the first term under the radical sign in (8) is generally negligible. Indeed, at a low heat-transfer rate (at low values of α and α_1) the second term under the radical sign is small. Under such conditions, as follows from (1), the temperature profile of a perforated fin insulated at the end will tend to become a uniform one ($\vartheta/\vartheta_0 \to 1$). According to (5), however, this is possible only at small values of μ, and this indicates that the first term is small. At finite values of α and α_1, therefore, only the second term in (8) is significant – the term which represents the mean-integral ratio of convective and conductive thermal conductances (or thermal resistances).

All this leads to physical considerations which are important to our problem. It is well known that the steady-state temperature profile of a solid thin fin is a function of the Biot number, which in this case represents the ratio of convective to conductive thermal conductance [7]:

$$\mathrm{Bi}=\frac{\alpha 2l}{\frac{\lambda}{l}\delta}. \tag{9}$$

As a result of perforations, the ratio of thermal conductances changes on account of the change in the surface area, in the transverse section, and in the heat-transfer rate at the fin elements. Accordingly, the temperature profile of a fin becomes distorted. A perforated fin can, as an approximation, be replaced by an equivalent solid fin with the same thermal conductances. The temperature profiles of such a fin and its perforated prototype must be identical. With the Biot number determined for the equivalent solid fin, the temperature profile of the latter and thus of the perforated fin can be calculated by well-known formulas for a solid fin. The value of the Biot number for the equivalent solid fin can be found by integrating the mean-over-the-height thermal conductances. We have

$$\mathrm{Bi}=\frac{2\alpha l^2}{\lambda\delta}\cdot\frac{b}{c}\left[1-\frac{\pi a^2}{4bc}+\frac{\alpha_1\pi a\delta}{2\alpha bc}\right]\left[\frac{2b}{\sqrt{b^2-a^2}}\,\mathrm{arctg}\sqrt{\frac{b+a}{b-a}}-\frac{\pi}{2}+\frac{c-a}{b}\right]. \tag{10}$$

Expression (10), which represents the ratio of mean-integral conductances, must correspond closely to the second term under the radical sign in (8), representing, as has been noted earlier, the mean-integral ratio of these thermal conductances. It is possible to show that a definite integral over a product (particular) of two functions (in our case the thermal conductances) does not differ much from the product (particular) of their integrals, if both functions can be represented by a sum of a constant (not equal to zero) term and a relatively small variable term. The latter requirement is obviously satisfied for the integrand of the second term in (8) when the holes are small.

All this leads to the conclusion that the temperature profile and the efficiency of a perforated fin can be calculated by the formulas

$$\frac{\vartheta}{\vartheta_0}=\frac{\mathrm{ch}\left[\sqrt{\mathrm{Bi}}\left(1-\frac{x}{l}\right)\right]}{\mathrm{ch}\sqrt{\mathrm{Bi}}} \tag{11}$$

and

$$\frac{\bar{\vartheta}}{\vartheta_0}=\frac{\mathrm{th}\sqrt{\mathrm{Bi}}}{\sqrt{\mathrm{Bi}}}\cdot\frac{b}{c}\left(\frac{2b}{\sqrt{b^2-a^2}}\,\mathrm{arctg}\sqrt{\frac{b+a}{b-a}}-\frac{\pi}{2}+\frac{c-a}{b}\right), \tag{12}$$

where Bi is determined according to (10).

We will now show that the simplified formulas (10)-(12) yield results sufficiently accurate for engineering purposes (with the available option of a more accurate solution, if necessary). For this purpose, we have checked our method experimentally by simulating the temperature field of a perforated fin in an electrolytic trough [8].

The model was a 10-times magnification of an element of a perforated fin with the dimensions l = 60 mm, b = 3.1 mm, and c = 3.0 mm. The lateral walls of the trough were made of acrylic glass, while the bottom was made of polished glass-Textolite sheet. Sixty discrete electrodes 10 mm wide were laid on the bottom by brush coating. They were supplied from an audio-signal generator at a 300 Hz frequency. The potential distribution in the electrolyte was measured with a vacuum-tube voltmeter. The resistance to heat transfer from the lateral fin surface was simulated by electrical resistors of definite ohmic values in the network. The perforations and the heat-transfer rate at their lateral surfaces were simulated by glass and copper cylinders, respectively (with appropriate resistors soldered on) 10 and 15 mm in diameter.

With a sufficient electrolyte depth (approximately 30 mm), which almost eliminated the effect of trough perimeter wetting, the measured value of Bi did not exceed by more than 2-3% its value calculated for a solid fin. The experimental and the theoretical values (the latter according to the simplified method) of the temperature (potential) profile are compared in Fig. 2 for three models of a fin with perforations 10 mm in diameter. The agreement between tested and calculated values is entirely satisfactory. The systematic discrepancy between both sets for the Biot number (not more than 10%) is largely due to the disproportionately magnified effect of wetting, because, as a result of perforation, the wetted perimeter of the model is magnified almost 1.5 times more. In all tests with models of perforated fins the potential at various points on any section did not deviate more than 2%. Therefore, the original assumption of a uniform temperature field in a perforated fin was very nearly valid as long as $a/b < 0.3\text{-}0.4$ and $\alpha_1 \leq \alpha$.

In conclusion, we note that the proposed method of calculating the temperature profile can, evidently, be used also for other perforation patterns.

NOTATION

- ϑ, ϑ_0 are the temperature along the fin and temperature at the fin base, respectively, above ambient;
- Bi is the Biot number;
- α, α_1 are the heat-transfer coefficients for the lateral fin surface and for the surface of holes, respectively;
- λ is the thermal conductivity;
- l, δ are the fin height and thickness;
- a is the hole diameter;
- b, c are the longitudinal and transverse perforation pitches.

LITERATURE CITED

1. E. R. Eckert and R. M. Dreick, Theory of Heat and Mass Transfer [Russian translation], Gosénergoizdat, Moscow–Leningrad (1961).
2. N. M. Matveev, Methods of Integrating Ordinary Differential Equations [in Russian], Vysshaya Shkola (1967).
3. G. Stoker, Nonlinear Oscillations in Mechanical and Electrical Systems [Russian translation], Moscow (1952).
4. M. D. O. Strett, Lame, Mathieu, and Related Functions in Physics and Engineering [in Russian], Izd. ONTI, Khar'kov–Kiev (1935).
5. N. V. MacLachlan, Theory and Applications of Mathieu Functions [Russian translation], IL, Moscow (1953).
6. E. T. Whittacker and G. N. Watson, Study Course in Modern Analysis [Russian translation], Vol. 2, Moscow (1963).
7. A. A. Gukhman, Introduction to the Theory of Similarity [in Russian], Vysshaya Shkola (1963).
8. Yu. D. Deniskin and I. F. Nekrasova, Application of the Simulation Technique to Solving Problems of Heat Conduction in Electronic Devices [in Russian], Moscow (1969).